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ON SOME SPECTRAL PROBLEMS ARISING IN DYNAMIC POPULATIONS

Antoine Henrot, El-Haj Laamri, Didier Schmitt

Institut Elie Cartan
Nancy Université - CNRS
B.P. 239, 54 506 Vandoeuvre-lès-Nancy, France

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This paper is dedicated to Michel Pierre who sparks off the mathematical career of the three authors.

Abstract. We study a spectral problem related to a reaction-diffusion model where preys and predators do not live on the same area. We are interested in the optimal zone where a control should take place. First, we prove existence of an optimal domain in a natural class. Then, it seems plausible that the optimal domain is localized in the intersection of the living areas of the two species. We prove this fact in one dimension for small sized domains.

1. Introduction.

This work stems from the talk given by Michel Langlais [10] during the Workshop "Partial Differential Equations and Applications" devoted to Michel Pierre's sixtieth birthday. It deals with Michel Pierre's favorite topics : reaction-diffusion systems and shape optimization. More precisely, we propose a partial answer to an open question raised in [10] regarding internal stabilisation for reaction-diffusion systems of the predator-prey model posed on non coincident spatial domains.

The predator-prey model presented in [10] describes the evolution of two species which, contrary to the classic models, live in two different domains and interact only in the intersection of these two domains, supposedly not empty. This is motivated by the fact that the observations show that the reproduction domain of the predators and their predation domain are often different.

For the sake of completeness and for the reader's convenience let us recall how one can derive the mathematical problem from describing or modeling the predator-prey model posed on non-coincident spatial domains. For more details, see [1], [4], [6] and its references.

Let $u = u(t, x)$ be the density of predators at time $t \geq 0$ and position $x \in \Omega_1$ and $v = v(t, y)$ the density of preys at time $t \geq 0$ and position $y \in \Omega_2$ where Ω_1 and Ω_2 are non-empty bounded open subsets of $\mathbb{R}^N$ with smooth boundaries and $\Omega_1 \cap \Omega_2 \neq \emptyset$.

We follow here [4] and the references therein. One assumes that in absence of prey the predator population will decay at a exponential rate $a(x)$, and diffuse with

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a constant diffusivity $d_1 > 0$. Then, one assumes that in absence of predators the dynamic of the prey population is governed by a logistic growth, with a natural growth rate $r(y) \geq 0$ and a density depending on the effect of mortality $k(y) > 0$ while preys diffuse with a constant diffusivity $d_2 > 0$.

As announced previously, interactions only occur in the common zone $\Omega_1 \cap \Omega_2$ through the quantity $\pi(u(t,y), v(t,y), y)$ which represents the number of preys killed by predators at time $t \geq 0$ and position $y \in \Omega_1 \cap \Omega_2$.

The biomass obtained with the killed preys is assumed to be transformed into birth rate through the parameter $\varepsilon > 0$. Nevertheless, since reproduction areas are generally distinct from predation areas, a function $\ell(x,y) \geq 0$ is introduced, which represents the biomass transfer from a point $y \in \Omega_1 \cap \Omega_2$ to a point $x \in \Omega_1$. To ensure the conservation of biomass, we need to impose

$$\int_{\Omega_1} \ell(x,y)dx = 1, \text{ for all } y \in \Omega_1 \cap \Omega_2$$

and that $\ell(x,y) = 0$ for any $y \in \Omega_2 \setminus \Omega_1$. Moreover, we assume that predators and preys are isolated in their respective areas and that at time $t = 0$, we have $u(0,x) = u_0(x)$ and $v(0,y) = v_0(y)$ with given data u_0 and v_0.

These considerations yield to the following reaction-diffusion system :

$$(\mathcal{S}) \begin{cases} \partial_t u - d_1 \Delta_x u = & \\ -a(x)u + \varepsilon \displaystyle\int_{\Omega_1 \cap \Omega_2} \ell(x,y)\pi(u(t,y), v(t,y), y)u(t,y)dy & t > 0, \quad x \in \Omega_1 \\ \partial_t v - d_2 \Delta_y v = & \\ (r(y) - k(y)v)v - \chi_{\Omega_1 \cap \Omega_2}(y)\pi(u(t,y), v(t,y), y)u(y) & t > 0, \quad y \in \Omega_2 \\ \dfrac{\partial u}{\partial \nu_1}(t,x) = 0 & t > 0, \quad x \in \partial\Omega_1 \\ \dfrac{\partial v}{\partial \nu_2}(t,y) = 0 & t > 0 \;\; y \in \partial\Omega_2 \\ u(0,x) = u_0(x) & x \in \Omega_1 \\ v(0,y) = v_0(y) & y \in \Omega_2, \end{cases}$$

where we denote by :
$\chi_{\Omega_1 \cap \Omega_2}$ the characteristic function of the intersection $\Omega_1 \cap \Omega_2$, $\nu_1(x)$ the exterior unit normal vector to $\partial\Omega_1$ at $x \in \partial\Omega_1$ and $\nu_2(y)$ the exterior unit normal vector to $\partial\Omega_2$ at $y \in \partial\Omega_2$.

It is proved in [1, Theorem 1] that the system $(\mathcal{S})$ admits a unique nonnegative global solution when the functions a, r and are constant and $\pi(u(t,y), v(t,y), y) = \dfrac{ev(t,y)}{1 + hev(t,y)}$ where h and e are positive real numbers.

In this work, we are interested in the large time stabilization of the predator population towards 0 while preserving a prey population, *i.e.* $u(t,.) \to 0$ and $v(t,.) \to v^*(.) > 0$ when $t \to +\infty$. To this end, a control localized in a small subdomain ω of Ω_1 can be introduced. Then, the equation describing the density of predators is modified into :

$$\partial_t u - d_1 \Delta u = -a(x)u - \gamma \chi_\omega u + \varepsilon \int_{\Omega_1 \cap \Omega_2} \ell(x,y) \frac{ev(t,y)}{1+hev(t,y)} u(t,y)dy, \qquad (t,x) \in (0,+\infty) \times \Omega_1.$$

It is well known that the asymptotic behaviour of solutions is closely connected to the sign and the magnitude of the first eigenvalue for the problem

$$\begin{cases} -d_1 \Delta u(x) + a(x)u(x) + \gamma u(x)\chi_\omega \\ \qquad - \displaystyle\int_{\Omega_1 \cap \Omega_2} \ell(x,y) \frac{ev^*(y)}{1+hev^*(y)} u(y)dy = \lambda_1(\omega)u(x) & x \in \Omega_1 \\ \qquad \dfrac{\partial u}{\partial \nu_1}(x) = 0 & x \in \partial\Omega_1. \end{cases}$$

Moreover, the greater $\lambda_1(\omega)$ is, the quicker the extinction of the predator population will be.

Therefore, the natural question is to investigate the existence of an admissible subdomain ω^* which maximizes $\lambda_1(\omega)$. And if the answer is yes, is it possible to specify its localization?

Intuitively, the optimal subdomain ω^* should be localized in the common zone $\Omega_1 \cap \Omega_2$. Indeed, it was shown by some numerical experiments presented by M. Langlais in his talk and confirmed by some simple computations which appear in section 5.

In this article, we are going to study these questions in a rather general framework. Since some of the basic ideas have an abstract character, we have chosen to first write some general properties and focus on the one-dimensional case only when needed. The general model is of the form

$$(\mathcal{P}) \begin{cases} -\Delta u + a(x)u + \gamma u \chi_\omega - \displaystyle\int_\Omega K(x,y)u(y)\,dy = \lambda_1(\omega)u & x \in \Omega \\ \qquad \dfrac{\partial u}{\partial \nu}(x) = 0 & x \in \partial\Omega, \end{cases}$$

where K is a non symmetric kernel *i.e* $K(x,y) \neq K(y,x)$ as explained in the above model.

Since the operator is not self-adjoint, the problem is more difficult to tackle. In particular, variational techniques using min-max formula cannot be used. Thus, a general study for the maximization problem of the first eigenvalue of $(\mathcal{P})$, following the lines of [5] (see also [7, chapter 8]) cannot be used here.

In section 2, we will prove the existence of a subdomain ω^* such as

$$\lambda_1(\omega^*) = \max_{\omega \in \mathcal{A}_{V_0,c}} \lambda_1(\omega)$$

where $\mathcal{A}_{V_0,c} := \{\omega \text{ measurable}\,;\;\; \omega \subset \Omega,\; |\omega| = V_0,\; |\partial\omega| \leq c\}$.

In section 3, we will prove the differentiability of the first eigenvalue $\lambda_1(\omega)$ with respect to the domain ω. In section 4 we will give some elements of answer to the second question in dimension 1. More precisely, under some natural assumptions on the kernel K, we prove that the optimal domain ω^* must intersect $\Omega_1 \cap \Omega_2$ when its length ℓ is small enough. This result is probably true without any assumption on the length. Finally, in the last section, we will give some perspectives.

2. **Existence of an optimal domain.**

Let us consider the problem

$$(\mathcal{P}_1)\begin{cases} -\Delta u(x) + a(x)u(x) + \gamma\chi_\omega(x)\,u & \\ \qquad\qquad -\displaystyle\int_\Omega K(x,y)u(y)\,dy + \zeta u(x) & = g(x) \quad x\in\Omega \\ \qquad\qquad\qquad\qquad \dfrac{\partial u}{\partial \nu}(x) & = 0 \qquad\; x\in\partial\Omega, \end{cases}$$

where Ω is a bounded regular (Lipschitz) open subset of $\mathbb{R}^N$, ω is a nonempty measurable set such that $\overline{\omega}\subset\Omega$, $(\gamma,\zeta)\in]0,+\infty[^2$, $a\in L^\infty(\Omega)$, $K\in L^\infty(\Omega\times\Omega)$ such that $a, K\geq 0$ a.e . Let us recall that the kernel K is not symmetric.
Assume that

$$1+\|a\|_{L^\infty(\Omega)}+\|K\|_{L^2(\Omega\times\Omega)}<\zeta. \tag{1}$$

For $(u,v)\in H^1(\Omega)\times H^1(\Omega)$, let us introduce the bilinear form:

$$\mathcal{A}_\omega(u,v)=\int_\Omega \nabla u\nabla v+\int_\Omega auv+\int_\Omega \gamma u\chi_\omega v-\int_\Omega\left(\int_\Omega K(x,y)u(y)\,dy\right)v(x)\,dx+\int_\Omega \zeta uv.$$

Thanks to (1), this bilinear form $\mathcal{A}_\omega$ is continuous and elliptic in $H^1(\Omega)$ (moreover the ellipticity constant can be chosen independently of ω). Then, Lax-Milgram theorem implies that for any $g\in L^2(\Omega)$, the problem (P_1) has a unique solution $\psi\in H^1(\Omega)$. Let us denote by T_ω the operator which maps every $g\in L^2(\Omega)$ into the solution $\psi=T_\omega(g)$ of (P_1) .

It is straightforward to check that $T_\omega : L^2(\Omega)\to L^2(\Omega)$ is a linear compact operator, with positive spectral radius $r(T_\omega)$. Moreover it satisfies $T_\omega(\mathcal{E})\subset\mathring{\mathcal{E}}$, where $\mathcal{E}:=\{\psi\in L^2(\Omega)\ ;\ \psi\geq 0 \text{ a.e in } \Omega\}$ and $\mathring{\mathcal{E}}$ is its interior.

Let us remark that, except if K is symmetric ($K(x,y)=K(y,x)$), the operator T_ω is not self-adjoint. Now since $\mathcal{E}$ is a solid cone, Krein-Rutman's theorem (see e.g [2, théorème VI.13]) yields that the spectral radius $r(T_\omega)$ is an eigenvalue of T_ω with an eigenfunction $u\in\mathcal{E}\setminus\{0\}$ such that $T_\omega(u)=r(T_\omega)u$. Moreover, $r(T_\omega^*)=r(T_\omega)$ is an eigenvalue of the adjoint operator T_ω^* with an eigenfunction $u^*\in\mathcal{E}\setminus\{0\}$.

It follows that $\lambda_1(\omega)=\dfrac{1}{r(T_\omega)}-\zeta$ is an eigenvalue for the following elliptic problem

$$(\mathcal{P}_2)\begin{cases} -\Delta u + a(x)u + \gamma u\chi_\omega - \displaystyle\int_\Omega K(x,y)u(y)\,dy & = \lambda_1(\omega)u \quad x\in\Omega \\ \qquad\qquad\qquad \dfrac{\partial u}{\partial\nu}(x) & = 0 \qquad\quad x\in\partial\Omega, \end{cases}$$

and $\lambda_1(\omega)$ is also an eigenvalue for the adjoint problem

$$(\mathcal{P}_2^*)\begin{cases} -\Delta u^* + a(x)u^* + \gamma u^*\chi_\omega - \displaystyle\int_\Omega K(y,x)u^*(y)\,dy & = \lambda_1(\omega)u^* \quad x\in\Omega \\ \qquad\qquad\qquad \dfrac{\partial u^*}{\partial\nu}(x) & = 0 \qquad\quad x\in\partial\Omega. \end{cases}$$

Moreover, Krein-Rutman's theorem shows that $\lambda_1(\omega)$ is a simple eigenvalue for $(\mathcal{P}_2)$ and $(\mathcal{P}_2^*)$ and there is no other eigenvalue with a positive eigenfunction.$\square$

Let V_0 and c be two positive real numbers such that $V_0 < |\Omega|$. We introduce the class of admissible domains

$$\mathcal{A}_{V_0,c} = \{\omega \text{ measurable}\,;\, \omega \subset \Omega\,,\, |\omega| = V_0\,,\, |\partial\omega| \leq c\}.$$

This set $\mathcal{A}_{V_0,c}$ is endowed with the topology associated to characteristic functions, namely: we say that a sequence $\{\omega_n\}_{n\geq 0}$ of sets in $\mathcal{A}_{V_0,c}$ converges *in the sense of characteristic functions* to $\omega \in \mathcal{A}_{V_0,c}$ if $\chi_{\omega_n} \to \chi_\omega$ in $L^p(\mathbb{R}^N)$, $\forall p \in [1, +\infty[$. According to Henrot and Pierre [8, théorème 2.3.10 and proposition 2.3.6], the set $\mathcal{A}_{V_0,c}$ equipped with the above topology is compact.

Theorem 2.1. *There exists $\omega^* \in \mathcal{A}_{V_0,c}$ such that*

$$\lambda_1(\omega^*) = \max_{w \in \mathcal{A}_{V_0,c}} \lambda_1(\omega).$$

To prove Theorem 2.1, it is sufficient to prove that the map $\omega \mapsto \lambda_1(\omega)$ from $\mathcal{A}_{V_0,c}$ to $\mathbb{R}$ is continuous. For this, we need the two following lemmas.

Lemma 2.2. *The map $\omega \mapsto T_\omega$ from $\mathcal{A}_{V_0,c}$ to $\mathcal{K}(L^2(\Omega))$ is continuous where $\mathcal{K}(L^2(\Omega))$ denotes the subspace of compact linear operators from $L^2(\Omega)$ into $L^2(\Omega)$.*
Proof.
Let $\{\omega_n\}$ be a sequence of elements of $\mathcal{A}_{V_0,c}$ which converges to ω in $\mathcal{A}_{V_0,c}$. We have to prove that $\{T_{\omega_n}\}$ converges to T_ω strongly *i.e*

$$\lim_{n\to+\infty} \|T_{\omega_n} - T_\omega\| = 0.$$

But according to Henrot [7, theorem 2.3.2], it suffices to prove that :
(i) the operators T_{ω_n} (and T_ω) are uniformly bounded,
(ii) for fixed $g \in L^2(\Omega)$, $\lim_{n\to+\infty} \|T_{\omega_n}(g) - T_\omega(g)\|_{L^2(\Omega)} = 0$.

In the following, we set $u_n = T_{\omega_n}(g)$ et $u = T_\omega(g)$.

The first point (i) comes immediately from uniform ellipticity (*i.e* with a constant independant of ω_n) of the bilinear forms $\mathcal{A}_{\omega_n}$, which implies

$$\|u_n\|_{H^1(\Omega)} \leq C\|g\|_{L^2(\Omega)} \qquad \text{with } C \text{ independent of } n. \tag{2}$$

Let us now prove (ii)

By virtue of (2), the sequence $\{u_n\}$ is bounded in $H^1(\Omega)$. Therefore, we can extract a subsequence, still denoted u_n, such that u_n converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some fonction w which belongs to $H^1(\Omega)$. We have to prove that $w = u$ and that the whole sequence converges to u.

For any integer n and for any $v \in H^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega \nabla u_n \nabla v + \int_\Omega a u_n v + \gamma \int_\Omega \chi_{\omega_n} u_n v \qquad & \\ - \int_\Omega v(x) \left(\int_\Omega K(x,y) u_n(y)\,dy \right) dx + \zeta \int_\Omega u_n v \quad &= \int_\Omega gv. \end{aligned} \tag{3}$$

We want to pass to the limit in the variational formulation (3).

By Rellich theorem, $H^1(\Omega)$ is compactly embedded in $L^p(\Omega)$ for any $p < \dfrac{2N}{N-2}$ if $N > 2$ (and for any $p < +\infty$ if $N = 2$). Thus we can assume that u_n converges strongly in $L^{2(N-1)/(N-2)}(\Omega)$ to w which implies that $u_n v$ converges strongly to wv in $L^{(N-1)/(N-2)}(\Omega)$. Since $\chi_{\omega_n} \to \chi_\omega$ in $L^{N-1}(\Omega)$ it follows that

$$\int_\Omega \chi_{\omega_n} u_n v \to \int_\Omega \chi_\omega w v.$$

Passing to the limit in the other terms is easy and we finally get that w satisfies

$$\begin{aligned}\int_\Omega \nabla w \nabla v + \int_\Omega awv + \gamma \int_\Omega \chi_\omega wv \\ - \int_\Omega v(x) \left(\int_\Omega K(x,y) w(y)\, dy \right) dx + \zeta \int_\Omega wv \quad = \int_\Omega gv.\end{aligned} \tag{4}$$

But (4) is exactly the variationnel formulation defining u, then $u = w$.

At last, since u is the only accumulation point of the sequence u_n, the whole sequence converges to u. □

Lemma 2.3. *The map $T_\omega \mapsto r(T_\omega)$ from $\mathcal{K}(L^2(\Omega))$ to $[0, +\infty[$ is continuous.*
Proof.
Lemma 2.3 is a particular case of the following general result :

Let E be a $\mathbb{K}$-Banach space where $K = \mathbb{R}$ or $\mathbb{C}$. Denote by $\mathcal{L}(E)$ its space of bounded linear operators and by $\mathcal{K}(E)$ the subspace of compact linear operators. Then with respect to the norm-operator the spectral radius mapping is upper semi-continuous on $\mathcal{L}(E)$ and continuous on $\mathcal{K}(E)$.

For a detailed proof, see Degla [3, Theorem 2.1]. But for the sake of completeness and for the reader's convenience, we sketch a proof going along the lines of Kato [9, paragraph 3.5.].

Let $\{T_n\}_n$ be a sequence in $\mathcal{K}(E)$ which converges to some $T \in \mathcal{L}(E)$. We are going to prove that

$$\limsup_{n\to+\infty} r(T_n) \le r(T) \le \liminf_{n\to+\infty} r(T_n).$$

First step : Let $\varepsilon, R \in (0, +\infty)$ such that $r(T) + \varepsilon < R$. Since $T_n \to T$ and thanks to Kato [9, theorem 3.1], the compact set $\{z \in \mathbb{C}\,;\ r(T) + \varepsilon < |z| < R\}$ is contained in the resolvent set of T_n (denoted $\rho(T_n)$) for n large enough. Therefore, there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$, we have

$$\{z \in \mathbb{C}\,;\ r(T) + \varepsilon < |z|\} \subset \rho(T_n)$$

and thus

$$\sigma(T_n) \subset \{z \in \mathbb{C}\,;\ |z| \le r(T) + \varepsilon\}$$

where $\sigma(T_n)$ denotes the spectrum of T_n. Hence $\limsup\limits_{n\to+\infty} r(T_n) \le r(T)$.

Second step : Since $\mathcal{K}(E)$ is a closed subspace of $\mathcal{L}(E)$, T is necessarily compact. If $r(T) = 0$,the result is clearly true. Assume that $r(T) > 0$. Using Krein-Rutman's Theorem, there exists an isolated eigenvalue λ of T such that $|\lambda| = r(T)$. Therefore, there exists $\varepsilon_0 > 0$ such that the circle $\mathcal{C}_0$ with center λ and radius ε_0 does not contain any other eigenvalue of T. According to Kato [9, paragraph 3.5] and since $T_n \to T$, for any $\varepsilon < \varepsilon_0$, $\exists n_0$ such that for $n \ge n_0$, the circle $\mathcal{C}_0$ contains an eigenvalue $\lambda_k(T_n)$ of T_n. But $\lambda_k(T_n) \le r(T_n)$. Then $r(T) \le \liminf\limits_{n\to+\infty} r(T_n)$. □

Remark 1 (**Remark on the perimeter constraint**)**.** If we remove the perimeter constraint, meaning that we work with the class

$$\mathcal{A}_{V_0} = \{\omega \text{ measurable} \subset \Omega\,,\ |\omega| = V_0\},$$

the problem becomes ill-posed. Indeed, we have the following result, in the particular case $a = K = 0$ which should extend to the general case.

Proposition 1. *Assume that $a = K = 0$, then $\sup\{\lambda_1(\omega)\,;\ \omega \in \mathcal{A}_{V_0}\} = \frac{\gamma V_0}{|\Omega|}$ and it is not achieved (meaning that the maximization problem has no solution).*

Proof.
Since $K=0$, the operator is self-adjont here and we have for every $\omega \in \mathcal{A}_{V_0}$

$$\lambda_1(\omega) = \min_{v \in H^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v| + \gamma \int_\Omega \chi_\omega v^2}{\displaystyle\int_\Omega v^2}.$$

Thus taking $v=1$, we obtain $\lambda_1(\omega) \le \frac{\gamma V_0}{|\Omega|}$. So that $\sup_{\omega \in \mathcal{A}_{V_0}} \lambda_1(\omega) \le \frac{\gamma V_0}{|\Omega|}$.
Let us show that in fact $\sup_{\omega \in \mathcal{A}_{V_0}} \lambda_1(\omega) = \frac{\gamma V_0}{|\Omega|}$.
According to Henrot and Pierre [8, Proposition 7.2.14], we can construct a sequence $\{\omega_n\}$ of elements of $\mathcal{A}_{V_0}$ such that χ_{ω_n} converges towards a constant ℓ in the weak-* topology. In this way,

$$\int_\Omega \chi_{\omega_n} = |\omega_n| = V_0 \to \int_\Omega \ell = \ell |\Omega|$$

and thus $\ell = \frac{V_0}{|\Omega|}$.
Now, according to Henrot [7, Theorem 8.1.2] (Continuity of $\lambda_1(V)$ for the weak-*convergence), we have

$$\lambda_1(\chi_{\omega_n}) = \lambda_1(\omega_n) \to \lambda_1\left(\frac{V_0}{|\Omega|}\right).$$

Moreover, $\lambda_1\left(\frac{V_0}{|\Omega|}\right)$ is the first eigenvalue of the following Neumann problem

$$\begin{cases} -\Delta u + \frac{\gamma V_0}{|\Omega|} u & = & \lambda u & \text{on } \Omega \\ \frac{\partial u}{\partial \nu} & = & 0 & \text{on } \partial\Omega \end{cases}$$

whence $\lambda_1\left(\frac{V_0}{|\Omega|}\right) = \frac{\gamma V_0}{|\Omega|}$.
Hence we have found a sequence $\{\omega_n\}$ such that $\lambda_1(\omega_n) \to \frac{\gamma V_0}{|\Omega|}$, proving the equality.
But for every $\omega \in \mathcal{A}_{V_0}$, we have $\lambda_1(\omega) < \frac{\gamma V_0}{|\Omega|}$. For if we had equality, then the function $u=1$ would necessarily be an eigenfunction for

$$\begin{cases} -\Delta u + \chi_\omega u & = & \lambda u & \text{on } \Omega \\ \frac{\partial u}{\partial \nu} & = & 0 & \text{on } \partial\Omega, \end{cases}$$

which is impossible. □

3. **Proof of differentiability of the eigenvalue.**

In this section, we prove differentiability of the first eigenvalue with respect to the domain. We follow notations of [8]. Let us consider a set $\omega \subset \Omega$ and an application Φ such that $\Phi : t \in [0,T[\to W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ is differentiable at 0 with $\Phi(0) = I$, $\Phi'(0) = V$, where $W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ is the set of bounded Lipschitz maps from $\mathbb{R}^N$ into itself, I is the identity and V is a deformation field. Let us denote by $\omega_t = \Phi(t)(\omega)$ and $\lambda_1(t)$ the first eigenvalue $\lambda_1(\Omega, \omega_t, \gamma)$ of problem $(\mathcal{P}_2)$ with ω replaced by ω_t. We will also denote by $u(t)$ a normalized eigenfunction associated to $\lambda_1(t)$ and by $u^*(t)$ a normalized eigenfunction associated to the adjoint problem $(\mathcal{P}_2^*)$. We are interested in the differentiability of the map $t \mapsto \lambda_1(t)$. The following Theorem states the differentiability and gives the formula of the first derivative when ω is Lipschitz.

Theorem 3.1. *Let λ_1, u and u^* denote respectively, the first eigenvalue, a normalized associated eigenfunction of problem $(\mathcal{P}_2)$ and a normalized eigenfunction associated to the adjoint problem $(\mathcal{P}_2^*)$.*
Then the map $t \mapsto \lambda_1(t)$ is differentiable at $t = 0$. Moreover, if ω is Lipschitz regular, the derivative is given by

$$\lambda_1'(0) = \gamma \frac{\int_{\partial\omega} uu^* V.\nu \, d\sigma}{\int_\Omega uu^*(x) \, dx} \tag{5}$$

where ν is the exterior unit normal vector to $\partial\omega$.
Proof.
In the proof we denote by H the Hilbert space $H^1(\Omega)$ and by H' its dual space. Let us introduce the operator $\mathcal{F}$ from $\mathbb{R} \times H \times \mathbb{R}$ with values in $H' \times \mathbb{R}$ and defined by the formula

$$\mathcal{F}(t, v, \lambda) = \big(-\Delta v + a(x)v + \gamma\chi_{\omega_t} v - \int_\Omega K(x,y)v(y)\,dy - \lambda v, \int_\Omega v^2 - 1 \big).$$

This operator is $\mathcal{C}^1$, the only difficulty being to prove differentiability of $t \mapsto \chi_{\omega_t} v$. But in H', $\chi_{\omega_t} v$ is the linear map $u \mapsto \int_\Omega \chi_{\omega_t} uv\,dx = \int_{\omega_t} uv\,dx$ which is $\mathcal{C}^1$ (and actually $\mathcal{C}^\infty$) according to [8, Theorem 5.2.2]. Moreover its derivative at $t = 0$ is given by

$$\frac{d}{dt}(\chi_{\omega_t} v) = \int_\omega \operatorname{div}(uvV)\,dx \tag{6}$$

which can be written, if ω is Lipschitz,

$$\frac{d}{dt}(\chi_{\omega_t} v) = \int_{\partial\omega} uvV.\nu \, d\sigma. \tag{7}$$

Now differentiating $\mathcal{F}$ with respect to its second and third variables yields, for any $(\hat{v}, \hat{\lambda}) \in H \times \mathbb{R}$:

$$D_{v,\lambda}\mathcal{F}(0, u, \lambda_1)(\hat{v}, \hat{\lambda}) = \big(-\Delta\hat{v} + a(x)\hat{v} + \gamma\chi_\omega\hat{v} - \int_\Omega K(x,y)\hat{v}(y)\,dy - \lambda_1\hat{v} - \hat{\lambda}u, 2\int_\Omega u\hat{v}\big).$$

Let us now prove that $D_{v,\lambda}\mathcal{F}(0, u, \lambda_1)$ is an isomorphism from $H \times \mathbb{R}$ onto $H' \times \mathbb{R}$. Since it is a continuous map, we have only to prove that it is one to one, thanks to the Banach open mapping Theorem. This is the purpose of the following Lemma.

Lemma 3.2. *Let $(Z, \Lambda) \in H' \times \mathbb{R}$ be given, then there exists a unique solution $(\hat{v}, \hat{\lambda}) \in H \times \mathbb{R}$ for the system*

$$\begin{cases} -\Delta\hat{v} + a(x)\hat{v} + \gamma\chi_\omega\hat{v} - \int_\Omega K(x,y)\hat{v}(y)\,dy\lambda_1\hat{v} - \hat{\lambda}u & = Z, \quad x \in \Omega \\ \dfrac{\partial\hat{v}}{\partial\nu}(x) & = 0, \quad x \in \partial\Omega \\ 2\int_\Omega u\hat{v} & = \Lambda\,. \end{cases} \tag{8}$$

Proof.
Of course, the system (8) has to be understood in the variational sense. By compactness of the operator $\left(-\Delta+a(x)I+\gamma\chi_\omega I\int_\Omega K(x,y).\,dy\right)^{-1}:H'\to H\subset H'$, we can apply Fredholm alternative to the operator $-\Delta+a(x)I+\gamma\chi_\omega I-\int_\Omega K(x,y).\,dy-\lambda_1 I$. Since λ_1 is a simple eigenvalue, by definition the kernel of this operator is of dimension 1 and its image is characterized by one relation of orthogonality with the corresponding eigenfunction. Applying this alternative to $Z+\hat{\lambda}u$ writes

$$0=<Z+\hat{\lambda}u,u>_{H'\times H}=\hat{\lambda}+<Z,u>\ .$$

This equality uniquely defines $\hat{\lambda}$. Now, any possible inverse image of $Z+\hat{\lambda}u$ by $-\Delta+a(x)I+\gamma\chi_\omega I-\int_\Omega K(x,y).\,dy-\lambda_1 I$ can be written v_0+su where v_0 is a particular one. But the relation $2\int_\Omega u\hat{v}=\Lambda$ implies

$$\Lambda=2\int_\Omega u(v_0+su)=2s+\int_\Omega uv_0\,.$$

Thus, s is also uniquely determined: this proves existence and uniqueness of a solution for system (8). □

We come back to the proof of Theorem 3.1. We can now apply the Implicit Function Theorem: there exists a map $t\mapsto(v(t),\lambda(t))\in H\times\mathbb{R}$ which is of class $\mathcal{C}^\infty$ on a neighborhood $\mathcal{V}$ of the origin and a neighborhood $\mathcal{O}$ of $(0,u,\lambda_1)$ in $\mathbb{R}\times H\times\mathbb{R}$ such that

$$v(0)=u,\quad \lambda(0)=\lambda_1,\quad \mathcal{F}^{-1}(\{0\})\cap\mathcal{O}=\{(t,v(t),\lambda(t));t\in\mathcal{V}\}.$$

Thus the map $t\to(v(t),\lambda(t))$ necessarily coincides with the continuous function $t\mapsto(u(t),\lambda_1(t))$ and the differentiability of $t\mapsto\lambda_1(t)$ and $t\mapsto u(t)$ is proved. In the same way, we can prove differentiability of $t\mapsto u^*(t)$.

Now we want to prove formula (5). For that purpose, we differentiate the variational formulation of $(\mathcal{P}_2)$. We denote by λ_1', u' and $u^{*\prime}$ the derivatives (at $t=0$) of $\lambda_1(t)$, $u(t)$ and $u^*(t)$ respectively. For any fixed $v\in H$, using formula (7) and the chain rule, we have

$$\begin{aligned}\int_\Omega\nabla u'.\nabla v\,dx+\int_\Omega a(x)u'v\,dx+\gamma\int_\omega u'v\,dx+\gamma\int_{\partial\omega}uvV.\nu\,d\sigma-\\ \int_\Omega\int_\Omega K(x,y)u'(y)v(x)\,dy\,dx=\lambda_1\int_\Omega u'v\,dx+\lambda'\int_\Omega uv\,dx\,.\end{aligned}\tag{9}$$

We will also use the variational formulation of the adjoint problem $(\mathcal{P}_2^*)$, for any $v\in H$:

$$\begin{aligned}\int_\Omega\nabla u^*.\nabla v\,dx+\int_\Omega a(x)u^*v\,dx+\gamma\int_\omega u^*v\,dx-\\ \int_\Omega\int_\Omega K(y,x)u^*(y)v(x)\,dy\,dx=\lambda_1\int_\Omega u^*v\,dx.\end{aligned}\tag{10}$$

Now, we choose $v=u^*$ in (9) and $v=u'$ in (10) and we substract the two equations to get

$$\int_\Omega uu^*(x)\,dx\lambda'(0)=\gamma\int_{\partial\omega}uu^*V.\nu\,d\sigma.\tag{11}$$

which is the desired formula and finishes the proof. □

In one dimension, when looking for a subset ω which is a single interval $(x_1, x_1+\ell)$, we will use in the next section the derivative of the first eigenvalue λ_1 with respect to x_1. The previous Theorem gives immediately the answer since derivative with respect to x_1 is equivalent to look at the derivative with respect to a translation of the domain. Thus, we need to apply formula (5) with $V.\nu = +1$ on the right side of the interval and $V.\nu = -1$ on the left side. Therefore, we have proved

Corollary 1. *In one dimension, when we consider an interval $\omega = (x_1, x_1+\ell)$, the derivative of the first eigenvalue with respect to x_1 is given by*

$$\lambda_1'(x_1) = \gamma \frac{uu^*(x_1+\ell) - uu^*(x_1)}{\int_\Omega uu^*(x)\, dx} . \square \tag{12}$$

4. Position of the optimal domain.

Intuitively the position of the optimal domain should be localized in the common zone for a large class of kernels K (see section 5). Nevertheless this result is not easy to prove in the general case. That is why we assume a to be constant and we study only the one-dimensional case with a "small" domain of control. Furthermore, in one dimension bounding the perimeter corresponds to bound the number of connected components, we shall thus study the case of a single interval.

Let $\Omega = (0,1)$ and

$$K(x,y) = \chi_{(0,\alpha)}(y)\hat{K}(x,y), \quad \alpha \in (0,1)$$

(this means that the common domain is $(0,\alpha)$) and let us assume that the control domain is $\omega = (z, z+\ell)$ with $z \in (0,1)$ and $\ell > 0$ small enough.

With this notations the eigenvalue problem and its adjoint problem can be written

$$(\mathcal{P}_\ell) \begin{cases} -u_\ell'' + au_\ell + \gamma\chi_{(z,z+\ell)}u_\ell - \displaystyle\int_0^1 K(x,y)u_\ell(y)dy = \lambda_{1,\ell}(z)u_\ell \quad x \in (0,1) \\ u_\ell'(0) = u_\ell'(1) = 0 \\ \|u_\ell\|_{L^2(0,1)} = 1, \end{cases}$$

$$(\mathcal{P}_\ell^*) \begin{cases} -(u_\ell^*)'' + au_\ell^* + \gamma\chi_{(z,z+\ell)}u_\ell^* - \displaystyle\int_0^1 K(y,x)u_\ell^*(y)dy = \lambda_{1,\ell}(z)u_\ell^* \quad x \in (0,1) \\ (u_\ell^*)'(0) = (u_\ell^*)'(1) = 0 \\ \|u_\ell^*\|_{L^2(0,1)} = 1. \end{cases}$$

where $\lambda_{1,\ell}(z), u_\ell$ and u_ℓ^* denote respectively the first eigenvalue, the normalized associated eigenfunction of problem $(\mathcal{P}_\ell)$ and the normalized associated eigenfunction of the adjoint problem $(\mathcal{P}_\ell^*)$.

For such a domain of control $\omega = (z, z+\ell)$ with a small size, we prove with a simple condition on $\hat{K}(x,y)$ that the optimal domain of control must have a non empty intersection with the common zone $(0,\alpha)$.

Theorem 4.1. *If the map $x \mapsto \hat{K}(x,y)$ is decreasing for all $y \in (0,\alpha)$, then there exists $\ell_0 > 0$ such that for all $\ell \in (0,\ell_0)$ the optimal interval $\omega^* = (z^*, z^*+\ell)$ satisfies $z^* \le \alpha$.*

Proof.

The proof of this result relies on the corollary 1 which ensures that for all $\ell > 0$

$$\frac{\partial \lambda_{1,\ell}}{\partial z}(z) = \gamma \frac{u_\ell u_\ell^*(z+\ell) - u_\ell u_\ell^*(z)}{\int_0^1 u_\ell u_\ell^*(x)dx}. \tag{13}$$

As it is not obvious to describe the variations of the map $x \mapsto u_\ell u_\ell^*(x)$, we first study the variations of the map $x \mapsto u_0 u_0^*(x)$ where u_0 and u_0^* are solutions of the limit problem with $\ell = 0$:

$$(\mathcal{P}_0)\begin{cases} -u_0'' + au_0 - \displaystyle\int_0^\alpha \hat{K}(x,y)u_0(y)dy = \lambda_{1,0}u_0 \qquad x \in (0,1) \\ u_0'(0) = u_0'(1) = 0 \\ \|u_0\|_{L^2(0,1)} = 1, \end{cases}$$

$$(\mathcal{P}_0^*)\begin{cases} -(u_0^*)'' + au_0^* - \chi_{(0,\alpha)}\displaystyle\int_0^1 \hat{K}(y,x)u_0^*(y)dy = \lambda_{1,0}u_0^* \quad x \in (0,1) \\ (u_0^*)'(0) = (u_0^*)'(1) = 0 \\ \|u_0^*\|_{L^2(0,1)} = 1. \end{cases}$$

First, we prove the uniform convergence on $(0,1)$ of $u_\ell u_\ell^*$ to $u_0 u_0^*$ and of $(u_\ell u_\ell^*)'$ to $(u_0 u_0^*)'$.

Lemma 4.2. *For all $z \in (0,1)$, we have*

$$\lim_{\ell \to 0} \|u_\ell u_\ell^* - u_0 u_0^*\|_\infty = 0, \quad \lim_{\ell \to 0} \|(u_\ell u_\ell^*)' - (u_0 u_0^*)'\|_\infty = 0. \tag{14}$$

Proof.
Let z in $(0,1)$ and let (ℓ_n) be a sequence converging to 0. In order to simplify notations, we write $u_n = u_{\ell_n}$ and $\lambda_{1,n} = \lambda_{1,\ell_n}(z)$.
Let us multiply the first equation of $(\mathcal{P}_n)$ by $1/\int_0^1 u_n$ and let us integrate on $(0,1)$. As we have $\int_0^1 \hat{K}(x,y)dx = 1$ for all $y \in (0,\alpha)$, we obtain

$$\lambda_{1,n} = a + \gamma \frac{\int_z^{z+\ell_n} u_n(x)dx}{\int_0^1 u_n} - \frac{\int_0^\alpha u_n(y)dy}{\int_0^1 u_n}.$$

Thus, the sequence $(\lambda_{1,n})$ is uniformly bounded

$$|\lambda_{1,n}| \leq a + \gamma + 1 \tag{15}$$

and up to a subsequence we can consider that $(\lambda_{1,n})$ converges to $\bar{\lambda}$.

Multiplying the first equation of $(\mathcal{P}_n)$ by u_n and integrating on $(0,1)$ yields

$$\int_0^1 (u_n')^2 \leq \|K\|_{L^2((0,1)\times(0,1))} + \lambda_{1,n}.$$

Then, using (15), we obtain that there exists a constant $C > 0$ independent of ℓ_n and z such that $\|u_n\|_{H^1} \leq C$. Using the classical compact embedding $H^1 \hookrightarrow L^\infty$, we immediately deduce that, up to a subsequence, (u_n) converges uniformly to $\bar{u}$. Note that $\bar{u}$ is nonnegative since for all $n \in \mathbb{N}$, u_n is nonnegative and $\|\bar{u}\|_{L^2(0,1)} = 1$.

We also have, for all n and for all $v \in H^1(0,1)$

$$\int_0^1 u_n'v' + a\int_0^1 u_n v + \gamma \int_z^{z+\ell} u_n v - \int_0^1 \left(\int_0^\alpha \hat{K}(x,y)u_n(y)dy\right)v(x)dx = \lambda_{1,n}\int_0^1 u_n v.$$

Passing to the limit, we obtain

$$\int_0^1 \bar{u}'v' + a\int_0^1 \bar{u}v - \int_0^1 \left(\int_0^\alpha \hat{K}(x,y)\bar{u}(y)dy\right)v(x)dx = \bar{\lambda}\int_0^1 \bar{u}v.$$

As $\bar{u}$ is nonnegative and $\|\bar{u}\|_{L^2(0,1)} = 1$, then necessarily $\bar{u} = u_0$ and $\bar{\lambda} = \lambda_{1,0}$. At last, since u_0 and $\lambda_{1,0}$ are respectively the only accumulation point of the sequences

(u_n) and $(\lambda_{1,n})$, the whole sequences converge to u_0 and $\lambda_{1,0}$ respectively.

In conclusion, we have for all $z \in (0,1)$

$$\lim_{\ell\to 0} \lambda_{1,n}(z) = \lambda_{1,0} \quad \text{and} \quad \lim_{\ell\to 0} \|u_\ell - u_0\|_\infty = 0.$$

Now, we have for all $x \in (0,1)$

$$\begin{aligned}
(u'_\ell - u'_0)(x) &= \int_0^x (u''_\ell - u''_0)(y)dy \\
&\leq a\int_0^x (u_\ell - u_0)(y)dy + \gamma\ell\|u_\ell\|_\infty \\
&\quad - \int_0^\alpha \left((u_\ell - u_0)(t)\int_0^x \hat{K}(y,t)dy\right)dt - \lambda_{1,\ell}(z)\int_0^x u_\ell(y)dy \\
&\quad + \lambda_{1,0}\int_0^x u_\ell(y)dy
\end{aligned}$$

thus

$$\|u'_\ell - u'_0\|_\infty \leq C\left(\ell + |\lambda_{1,n} - \lambda_{1,0}| + \|u_\ell - \bar{u}\|_\infty\right).$$

which gives the uniform convergence of (u'_ℓ) to $(\bar{u}')$.
By a similar computation, we obtain the uniform convergence of (u^*_ℓ) and $(u^*_\ell)'$ respectively to u^*_0 and $(u^*_0)'$ and the result follows. □

Now, we study the limit problem and we have the following result :

Lemma 4.3. *If the map $x \mapsto \hat{K}(x,y)$ is decreasing for all $y \in (0,\alpha)$, then there exists two positive constants C_0 and θ such that*

$$(u_0 u^*_0)'(x) < C_0 sh(\theta(x-1)) \quad \forall x \in (\alpha, 1).$$

Proof.
First, note that on $(\alpha,1)$, u^*_0 is solution of the equation

$$-(u^*_0)''(x) + \theta^2 u^*_0(x) = 0, \quad (u^*_0)'(1) = 0$$

with $\theta^2 = a - \lambda_{1,0}$. Thus, there exists $A > 0$ such that for all $x \in (\alpha,1)$

$$u^*_0(x) = A\mathrm{ch}(\theta(1-x)).$$

Note also, that u_0 satisfies on $(0,1)$ the following equation

$$-u''_0(x) + \theta^2 u_0(x) = f(x), \quad u'_0(0) = u'_0(1) = 0,$$

where

$$f(x) = \int_0^\alpha \hat{K}(x,y)u_0(y)dy.$$

So, we have for all $x \in (0,1)$

$$u_0(x) = \frac{\mathrm{ch}(\theta x)}{\theta\mathrm{sh}(\theta)}\int_0^1 \mathrm{ch}(\theta(1-t))f(t)dt - \frac{1}{\theta}\int_0^x \mathrm{sh}(\theta(x-t))f(t)dt.$$

Then, we deduce that for all $x \in (\alpha,1)$

$$\begin{aligned}
(u_0 u^*_0)'(x) &= \frac{A}{\mathrm{sh}(\theta)}\left[\mathrm{sh}(\theta(2x-1))\int_x^1 \mathrm{ch}(\theta(1-t))f(t)dt\right. \\
&\quad \left. - \mathrm{sh}(2\theta(1-x))\int_0^x \mathrm{ch}(\theta t)f(t)dt\right].
\end{aligned}$$

Now, since $x \mapsto \hat{K}(x,y)$ is assumed to be decreasing for all $y \in (0,\alpha)$, the map $x \mapsto f(x)$ is also decreasing and then we have for all $x \in (\alpha, 1)$

$$\begin{aligned}(u_0 u_0^*)'(x) &\leq \frac{Ag(x)}{\theta \mathrm{sh}(\theta)}[\mathrm{sh}(\theta(2x-1))\mathrm{sh}(\theta(1-x)) - \mathrm{sh}(2\theta(1-x))\mathrm{sh}(\theta x)] \\ &\leq \frac{Ag(x)}{\theta}\mathrm{sh}(\theta(x-1))\end{aligned}$$

where

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \geq 1/2 \\ f(1) & \text{if} \quad x \leq 1/2. \end{cases}$$

Finally, we obtain

$$(u_0 u_0^*)'(x) < C_0 \mathrm{sh}(\theta(x-1)) \quad \forall x \in (\alpha, 1)$$

where C_0 is a positive constant. □

We are now in position to prove Theorem 4.1. By lemma 4.2, $(u_\ell u_\ell^*)$ converges uniformly to $u_0 u_0^*$, and lemma 4.3 gives the sign of $u_0 u_0^*(z+\ell) - u_0 u_0^*(z)$. However to obtain the sign of the application $u_\ell u_\ell^*(z+\ell) - u_\ell u_\ell^*(z)$ up to $x=1$, we have to cut the interval $[\alpha, 1-\ell]$ in two parts $[\alpha, 1-\beta]$ and $[1-\beta, 1-\ell]$.

Let $C_1 = \|(u_0 u_0^*)'\|_\infty + 1$ and let $\beta > 0$ small enough such that

$$\frac{C_0}{\theta}\left[\mathrm{ch}\left(\theta(1-\alpha-\beta/2)\right) - \mathrm{ch}(\theta\beta)\right] \geq \beta C_1 + C_0(1-\alpha)\mathrm{sh}(\theta\beta/2). \tag{16}$$

Lemma 4.2 ensures the uniform convergence of $((u_\ell u_\ell^*)')$ to $(u_0 u_0^*)'$, so there exists $\ell_1 > 0$ such that for all $\ell \in (0, \ell_1)$,

$$\|(u_\ell u_\ell^*)'\|_\infty \leq C_1 \tag{17}$$

and there exists $\ell_2 > 0$ such that for all $\ell \in (0, \ell_2)$

$$\|(u_\ell u_\ell^*)' - (u_0 u_0^*)'\|_\infty \leq C_0 \mathrm{sh}(\theta\beta/2)$$

which gives combining with lemma 4.3

$$(u_\ell u_\ell^*)'(x) \leq C_0 \mathrm{sh}(\theta(x-1)) + C_0 \mathrm{sh}(\theta\beta/2), \tag{18}$$

for all $\ell \in (0, \ell_2)$ and for all $x \in (\alpha, 1)$.

Now, we choose $\ell_0 = \min(\ell_1, \ell_2, \beta/2)$ and let $\ell \in (0, \ell_0)$.

¿From (13), we have for all $(z, z') \in [\alpha, 1-\ell]^2$

$$\begin{aligned}\lambda_{1,\ell}(z') - \lambda_{1,\ell}(z) &= \frac{1}{\int_0^1 u_\ell u_\ell^*(x)dx}\int_z^{z'} u_\ell u_\ell^*(y+\ell) - u_\ell u_\ell^*(y)dy \qquad (19) \\ &= C(\ell)\int_z^{z'}\int_y^{y+\ell}(u_\ell u_\ell^*)'(t)dtdy \qquad (20)\end{aligned}$$

Thus, for all $z \in (\alpha, 1-\beta)$ we have

$$\lambda_{1,\ell}(z) - \lambda_{1,\ell}(\alpha) = C(\ell)\int_\alpha^z \int_y^{y+\ell}(u_\ell u_\ell^*)'(t)dtdy$$

and as $z + \ell < z + \ell_0 < z + \beta/2$, we deduce immediately using (18)

$$\lambda_{1,\ell}(z) \leq \lambda_{1,\ell}(\alpha), \quad \text{for all } z \in (\alpha, 1-\beta).$$

The estimate on $[1-\beta, 1-\ell]$ is more difficult to obtain. By (20), we have for all $z \in [1-\beta, 1-\ell]$

$$\lambda_{1,\ell}(z) - \lambda_{1,\ell}(1-\beta) = C(\ell)\int_{1-\beta}^{z}\int_{y}^{y+\ell}(u_\ell u_\ell^*)'(t)dtdy$$

and then, using (17), we obtain for all $z \in [1-\beta, 1-\ell]$

$$\lambda_{1,\ell}(z) \leq \lambda_{1,\ell}(1-\beta) + C(\ell)\beta C_1 \ell. \tag{21}$$

By (20), we also have

$$\lambda_{1,\ell}(\alpha) - \lambda_{1,\ell}(1-\beta) = -C(\ell)\int_{\alpha}^{1-\beta}\int_{y}^{y+\ell}(u_\ell u_\ell^*)'(t)dtdy$$

and using (18), we deduce

$$\begin{aligned}\lambda_{1,\ell}(\alpha) - \lambda_{1,\ell}(1-\beta) &\geq -C(\ell)\int_{\alpha}^{1-\beta}\int_{y}^{y+\ell}\left(C_0\mathrm{sh}(\theta(t-1)) + C_0\mathrm{sh}(\theta\beta/2)\right)dtdy \\ &\geq -C(\ell)\ell(1-\beta-\alpha)C_0\mathrm{sh}(\theta\beta/2) \\ &\quad + \frac{C(\ell)C_0\ell}{\theta}\left[\mathrm{ch}(\theta(1-\alpha-\ell)) - \mathrm{ch}(\theta(\ell-\beta))\right].\end{aligned}$$

As $\ell \leq \ell_0 \leq \beta/2$, we deduce that

$$\begin{aligned}\lambda_{1,\ell}(\alpha) - \lambda_{1,\ell}(1-\beta) \geq \quad & \ell C(\ell)\left[-(1-\alpha)C_0\mathrm{sh}(\theta\beta/2)\right. \\ & \left. + \tfrac{C_0}{\theta}\left[\mathrm{ch}(\theta(\alpha-1-\beta/2)) - \mathrm{ch}(\theta\beta)\right]\right].\end{aligned}$$

Finally, using (16) and (21) we get $\lambda_{1,\ell}(z) \leq \lambda_{1,\ell}(\alpha)$, for all $z \in [1-\beta, 1-\ell]$.

In conclusion, we have for all $z \in [\alpha, 1-\ell]$

$$\lambda_{1,\ell}(z) \leq \lambda_{1,\ell}(\alpha).$$

□

5. **Conclusion.**

Some numerical simulations confirm Theorem 4.1 for larger values of ℓ. The Figure 1 presents the graph of $z \mapsto \lambda_1(z)$ for $\ell = 0.2$ with the following choices of the kernel K:

- (**left**): $K(x,y) = 4\chi_{(0,1/4)}(y)$,
- (**right**): $K(x,y) = (2(1-x)\cos^2(y) + (e/(e-1))\exp(-x)\sin^2(y))\chi_{(0,1/4)}$.

As we can see, the optimal interval is completely on the left in the first case, while it is shifted to the right in the second case but stays inside the common zone $\Omega_1 \cap \Omega_2$.

We have been able to prove that the optimal domain is in the common zone $\Omega_1 \cap \Omega_2$ only under the three following assumptions:

- dimension one
- interval of small size
- the kernel $K(x,y)$ is decreasing with respect to x.

It would be interesting to extend this result to intervals of arbitrary size and possibly for more general kernels. Nevertheless, we think that some assumptions on the kernel are needed to get such a property of the optimal domain. At last, it seems us challenging to study the same question in higher dimension.□

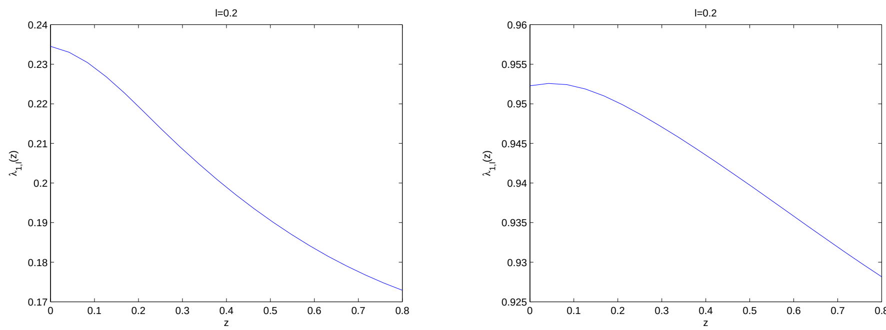

FIGURE 1. Graph of $z \mapsto \lambda_1(z)$

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E-mail address: Antoine.Henrot@iecn.u-nancy.fr
E-mail address: Elhaj.Laamri@iecn.u-nancy.fr
E-mail address: Didier.Schmitt@iecn.u-nancy.fr